NOTIZEN

Methods and Results of a General Theory of Particles with Spin and its Connection with SU_{12}

L. Castell *

School of Mathematics, Trinity College, Dublin

(Z. Naturforschg. **20 a**, 737—738 [1965]; received 26 March 1965)

Some recent attempts [1] to combine the charge symmetries of elementary particles i. e. SU_3 with their space-time structure make it desirable to review old results [2] and to point out some later developments [3, 4] in the theory of field equations for particles with arbitrary spin.

The most general linear, Lorentz invariant field equation of first order is given by

$$\left(\Gamma_\mu \frac{\partial}{\partial x_\mu} + \varkappa\right) \psi(x) = 0, \quad \mu = 1, 2, 3, 4. \qquad (1)$$

From the invariance of (1) under the homogeneous Lorentz transformations follow the commutation relations

$$[\beta_{\mu\nu}, \beta_{\varrho\sigma}] = i(\delta_{\mu\varrho}\,\beta_{\nu\sigma} - \delta_{\mu\sigma}\,\beta_{\nu\varrho} - \delta_{\nu\varrho}\,\beta_{\mu\sigma} + \delta_{\nu\sigma}\,\beta_{\mu\varrho}), \quad (2)$$

$$[\Gamma_\mu, \beta_{\varrho\sigma}] = i(-\delta_{\mu\varrho}\,\Gamma_\sigma + \delta_{\mu\sigma}\,\Gamma_\varrho), \qquad (3)$$

$$[\beta_{\mu\nu}, \varkappa] = 0 .$$

Here the matrices $\beta_{\mu\nu} = -\beta_{\nu\mu}$ are the generators of the homogeneous Lorentz transformations acting on the wave function ψ.

The equations (2) define the Lie algebra ${}^4O$. Now it has been shown by Bauer that the smallest semi-simple Lie algebra containing elements satisfying (2) and (3) is given by ${}^5O$, $\varkappa$ being proportional to the unit matrix. All other such algebras are extensions of ${}^5O$ e. g. ${}^6O$, ${}^7O$, etc. In the case [5] of ${}^5O$ the mass ratio of different particle states contained in each irreducible representation ${}^5O(m_1\, m_2)$ are completely determined. The higher algebras, which couple different equations of ${}^5O$ do not determine a mass spectrum without further assumptions. In general $\varkappa$ may be an arbitrary non singular function of the elements of the Lie algebra commuting with $\beta_{\mu\nu}$ [6].

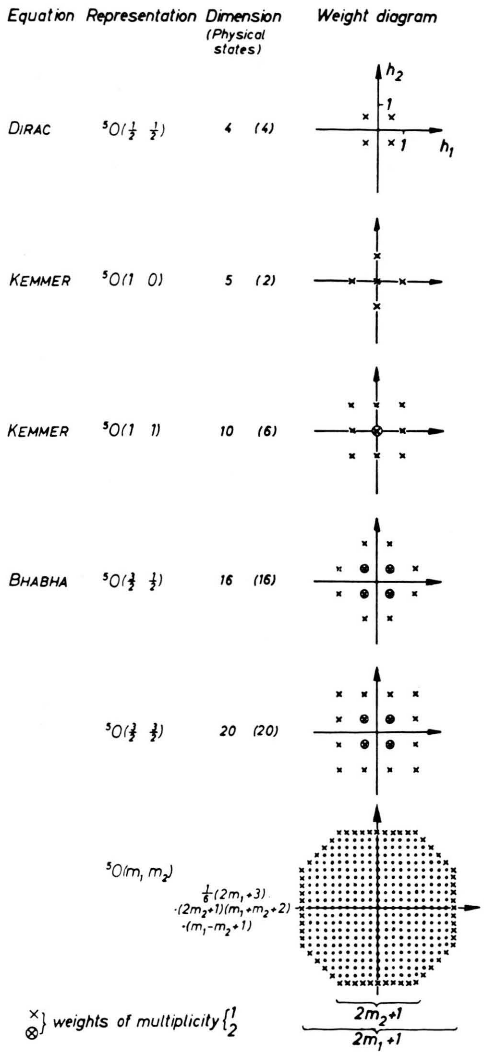
In all cases we can choose

$$\Gamma_\mu = 2\,\beta_{5\mu} = -2\,\beta_{\mu 5}$$

and the commutation relations (2) and (3) have always to be completed by

$$[\Gamma_\mu, \Gamma_\nu] = 4\,i\,\beta_{\mu\nu} .$$

Every finite dimensional irreducible representation ${}^5O(m_1\, m_2)$ $(m_1 \geqq m_2 \geqq 0$ both integer for tensor representations or half integer for spinor representations) uniquely determines a field equation of type (1). These equations have been discussed by many authors [2] but we shall present their results in a new form and point



* Research sponsored by the European Office of Aerospace Research, United States Air Force, Grant No. AF EOAR 64-61.

[1] R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. Letters; to be published; and Proc. Roy. Soc., Lond. A **284**, 146 [1965]. — P. Roman and J. J. Aghassi, Phys. Letters **14**, 68 [1965].

[2] H. Umezawa, Quantumt Field Theory, North-Holland Publishing Co., Amsterdam 1956.

[3] F. L. Bauer, Sitzb. d. Bay. Akad. d. Wiss. **13**, 111 [1952].

[4] L. Castell, D. I. C. Thesis, London 1961; Diplomarbeit, München 1962.

[5] J. K. Lubanski, Physica **9**, 310 [1942].

[6] F. Bopp and F. L. Bauer, Z. Naturforschg. **4 a**, 611 [1949].

out the connection with the BARGMANN–WIGNER equations, which have recently been used by DELBOURGO, SALAM, and STRATHDEE.

In the rest system equation (1) becomes

$$(-E\,\Gamma_4+\varkappa)\,\psi=0\,.$$

Note that the matrices $S_3=\beta_{12}$ (spin component in 3rd direction) and $\frac{1}{2}\,\Gamma_4=\beta_{54}$ may be chosen as the commuting set of elements H_1 and H_2 in the canonical form of the commutation relations. Each weight $(h_1\,h_2)$ of an irreducible representation determines then a particle state with a spin component $S_3=h_1$ and a rest energy $E=\varkappa/2\,h_2$, the weights characterized by $h_2=0$ being excluded (redundant states).

We exhibit some examples in Fig. 1.

As far as the free field equations are concerned one can choose as an initial condition one special mass value. The irreducible representations which are contained in the BARGMANN–WIGNER [7] equations are specified by the lowest values of the rest energy or by $h_2=\pm N/2$. This follows from the fact that the BARGMANN–WIGNER equations (without the symmetry condition on ψ)

$$\begin{aligned}
&\left(\gamma_\mu^{(1)}\frac{\partial}{\partial x_\mu}+m\right)\psi=0\,, \qquad \gamma_\mu^{(1)}=\gamma_\mu\times 1\times 1\times\ldots\times 1\,,\\
&\left(\gamma_\mu^{(2)}\frac{\partial}{\partial x_\mu}+m\right)\psi=0\,, \qquad \gamma_\mu^{(2)}=1\times\gamma_\mu\times 1\times\ldots\times 1\,, \qquad (4)\\
&\qquad\vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
&\left(\gamma_\mu^{(N)}\frac{\partial}{\partial x_\mu}+m\right)\psi=0\,, \qquad \gamma_\mu^{(N)}=1\times 1\times 1\times\ldots\times\gamma_\mu\,.
\end{aligned}$$

are equivalent to two equations

$$\left(\Gamma_\mu\frac{\partial}{\partial x_\mu}+N\,m\right)\psi=0\,, \qquad (5_1)$$

$$\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu}-m^2\right)\psi=0\,, \qquad (5_2)$$

with
$$\Gamma_\mu=\sum_{n=1}^{N}\gamma_\mu^{(n)}\,,$$

which can be seen easily by computing all plane wave solutions of (4) and (5). Equation (5_1) is nothing else than the reducible representation $[{}^5O(\frac{1}{2}\,\frac{1}{2})]^N$ of equation (1), and the initial condition (5_2) selects the lowest energy values. We are thus led to the result that the BARGMANN–WIGNER subset contained in each irreducible representation ${}^5O(m_1\,m_2)$ is characterized by the rows $h_2=\pm N/2$ of the corresponding weight diagram. These weights are simple if they exist and characterize a representation ${}^4O(m_2\,m_2)+{}^4O(m_2\;-m_2)$ of the subalgebra ${}^4O$. This representation describes a spinor in the DIRAC–FIERZ–PAULI theory. Therefore the set of equations (4) decomposes completely into single particle equations.

A further difference between (1) and (4) is that for $m_1\geqq 3/2$ equation (1) can be quantized only within the framework of an indefinite metric in HILBERT space. If we however separate off the lowest energy eigen values, as we have done in equation (4) a quantization according to the canonical formalism with positive definite metric becomes possible.

Let me now point out the differences and the similarities in the two theories quoted in reference [1]. The fundamental representation out of which DELBOURGO, SALAM, and STRATHDEE construct the mesons and the baryons by a "methode de fusion" is the quark Q

$$Q={}^6O(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2})\times SU_3(100)$$

with baryon number $\frac{1}{3}$. The anti-quark is given by Q^*

$$Q^*={}^6O(\tfrac{1}{2}\,\tfrac{1}{2}\;-\tfrac{1}{2})\times SU_3(00\;-1)\,.$$

ROMAN and AGHASSI restrict the space-time symmetry to ${}^5O$ so that

$$Q={}^5O(\tfrac{1}{2}\,\tfrac{1}{2})\times SU_3(100)\,,\quad Q^*={}^5O(\tfrac{1}{2}\,\tfrac{1}{2})\times SU_3(00\;-1)\,.$$

However this restriction does not lead to any difference as far as the particle states are concerned, because the restriction of ${}^6O(\frac{1}{2}\,\frac{1}{2}\,\pm\frac{1}{2})$ to ${}^5O$ is ${}^5O(\frac{1}{2}\,\frac{1}{2})$:

$${}^6O(\tfrac{1}{2}\,\tfrac{1}{2}\,\pm\tfrac{1}{2})\supset{}^5O(\tfrac{1}{2}\,\tfrac{1}{2})\,.$$

The LIE algebras of ${}^6O$ and SU_4 are isomorphic. The representations are connected by

$$\begin{aligned}{}^6O(\tfrac{1}{2}(f_1+f_2-f_3-f_4);\ \tfrac{1}{2}(f_1-f_2+f_3-f_4);\ \tfrac{1}{2}(f_1-f_2-f_3+f_4))\\ =SU_4(f_1;\,f_2;\,f_3;\,f_4)\,.\end{aligned}$$

The combination of SU_4 with SU_3 leads to SU_{12} under which all very strong interaction terms are assumed to be invariant. The quark and anti-quark are finally given by

$$\begin{aligned}Q&=SU_{12}(100\ldots0)\supset SU_4(1000)\times SU_3(100)\,,\\ Q^*&=SU_{12}(0\,0\ldots0\;-1)\supset SU_4(000\;-1)\times SU_3(00\;-1)\,.\end{aligned}$$

All mesons are now contained in the product

$$\begin{aligned}&Q^*\times Q=SU_{12}(100\ldots0\;-1)+SU_{12}(00\ldots0)\supset\\ &SU_4(100\;-1)\times SU_3(10\;-1)+SU_4(100\;-1)\\ &\times SU_3(000)+SU_4(0000)\times SU_3(10\;-1)+SU_4(0000)\\ &\times SU_3(000)\,,\end{aligned}$$

which decomposes into spin 1^- vector meson and a spin 0^- meson singlet and octet. The condition (5_2) does not give rise to any restriction in the case $N=2$. The baryons are given by entirely symmetric representation [3] contained in

$$\begin{aligned}Q\times Q\times Q=SU_{12}(300\ldots)+2\,SU_{12}(210\ldots)\\ +SU_{12}(1110\ldots.)\end{aligned}$$

for which we have

$$\begin{aligned}SU_{12}(30\ldots.)\supset SU_3(3000)\times SU_3(300)\\ +SU_4(2100)\times SU_3(10\;-1)\\ +SU_4(000\;-1)\times SU_3(000)\,.\end{aligned}$$

DELBOURGO, SALAM, and STRATHDEE restrict themselves now to the BARGMANN–WIGNER equations by applying (5_2) and obtain a spin $3/2^+$ decuplet and a spin $1/2^+$ octet. ROMAN and AGHASSI do not seem to apply condition (5_2) and obtain far more particle states.

Acknowledgement

The author would like to thank Dr. D. LURIÉ and Dr. D. SIMMS for inspiring discussions. He also expresses his thanks to the United States Air Force for a Research Fellowship.

[7] V. BARGMANN and E. P. WIGNER, Proc. Nat. Acad. Sci. U. S. **34**, 211 [1948]. — L. DE BROGLIE, Théorie générale des particules à spin (Méthode de fusion), Gauthier-Villars, Paris 1943.